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Effective action approach to the Leggett's mode in two-band superconductors

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Abstract. We investigate a collective excitation (Leggett's mode) corresponding to small fluctuations of the relative phase of two condensates in two-band superconductor using the effective "phase only" action. We consider the possibility of observing Leggett's mode in MgB_2 superconductor and conclude that for the known at present values of the two-band model parameters for MgB_2 Leggett's mode arises above the two-particle threshold.

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1 Introduction

The problem of collective modes in superconductors is almost as old as the microscopic theory of superconductivity. Bogolyubov [1] and Anderson [2] discovered that density oscillations can couple to oscillations of the phase of the superconducting order parameter *via* the pairing interaction. In a neutral system these collective soundlike oscillations are called Bogolyubov-Anderson-Goldstone (BAG) mode. In a charged system the frequency of this mode is pushed up to the plasma frequency due to the long-range Coulomb interaction [3].

Despite the fact that the basic physics of the BAG and plasma modes in superconductors is well understood many years ago, it appears that a more modern approach provides even better insight to the origin and universal character of these modes allowing also to tackle less settled questions. A main idea beyond this approach is rather simple: since the collective modes present usually low energy degrees of freedom, to study them it is sufficient to have only an effective Lagrangian (or action) that describes low-frequency, low-wavelength dynamics of the phase θ of the pairing field instead of working with the original Hamiltonian of the system. (Note that although plasmons are in general high-energy excitations, they still can be treated using an effective theory [4].)

The most convenient way of deriving such a Lagrangian is to change the variables (see, for example, [4]) for the complex pairing field $\Phi \to \Delta \exp(\mathrm{i}\theta)$ with real modulus Δ and phase θ. The simultaneous transformation for the fermi field $\psi_\sigma \to \chi_\sigma \exp(\mathrm{i}\theta/2)$ allows from the beginning to separate in the Hamiltonian the only phase degree of freedom relevant for the effective theory. In the theory of superconductivity this replacement of the variables that resembles a gauge transformation has, in fact, a long history and it was probably firstly used in [5] and then in [6] (see also Refs. in the review [7]). More recently this transformation was used by many authors. For example, in [8] it was used to study the problem of the Galilean invariance of the effective Lagrangian at $T = 0$. The finite temperature time-dependent effective actions for the phase field in s-wave [9] and d-wave [10] *neutral* superconductors were derived addressing an old problem of time-dependent generalization of the GL theory. It is also convenient to investigate the plasma mode within this formalism taking into account the long-distance Coulomb repulsion between electrons [4,6] (see also more recent papers [11,12]).

Besides these rather commonly known modes there are also other collective modes such as Carlson-Goldman mode [13] which can appear when the Coulomb interaction gets screened and Leggett's mode [14] which is present in two-band superconductors. Recently Carlson-Goldman mode has been studied in d-wave superconductors [15] and colour superconductivity [16] using the approach based on the modulus-phase variables, but to the best of our knowledge Leggett's mode has not been considered yet using this transformation. Thus it would be interesting and instructive to apply this method to obtain Leggett's mode.

Physically Leggett's mode [14] is a collective excitation corresponding to small fluctuations of the relative phase

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of two condensates in a clean two-band superconductor. There are some indications that MgB_2 superconductor with $T_c = 39\,\mathrm{K}$ [17] discovered about one year ago is an example of such a two-band system. Thus the main results of the present paper are two-fold and can be summarized as follows.
i) Leggett's mode is obtained using the modulus-phase variables in the path integral formalism;
ii) It is considered whether this mode can be observed in MgB_2 superconductor.

The paper is organized as follows. In Section 2 we introduce the two-band model and represent the partition function of the system using the Hubbard-Stratonovich transformations for pairing and Coulomb parts of the interactions. In Section 3 we express the effective thermodynamical potential of the system in the modulus-phase variables and obtain the system of equations for the superconducting gaps. Then we separate the part which describes the collective phase modes. In Sections 4 and 5 we derive the dispersion law for Leggett's mode in neutral and charged systems, respectively. Section 6 we investigate the possibility of observing Leggett's mode in MgB_2 superconductor. We conclude in Section 7 with a discussion and summary of our results.

2 Model two-band Hamiltonian and Hubbard-Stratonovich transformations

Let us consider the following action (in our notations the functional integral is expressed *via* e^S)

$$S = -\int_0^\beta \mathrm{d}\tau \left[\sum_{i=1}^{2}\sum_\sigma \int \mathrm{d}^2 r \psi^\dagger_{i\sigma}(\tau,\mathbf{r})\partial_\tau \psi_{i\sigma}(\tau,\mathbf{r}) + H(\tau)\right],$$
$$\mathbf{r} = (x,y,z), \qquad \beta \equiv \frac{1}{T}, \tag{1}$$

where the Hamiltonian $H(\tau)$ is

$$\begin{aligned} H(\tau) = &\sum_{i=1}^{2}\sum_\sigma \int \mathrm{d}^2 r \psi^\dagger_{i\sigma}(\tau,\mathbf{r})[\varepsilon_i(-\mathrm{i}\nabla) - \mu]\psi_{i\sigma}(\tau,\mathbf{r}) \\ &- \frac{1}{2}\sum_{i,j=1}^{2}\sum_\sigma \int \mathrm{d}\mathbf{r}_1 \int \mathrm{d}\mathbf{r}_2 \\ &\times \psi^\dagger_{i\sigma}(\tau,\mathbf{r}_2)\psi^\dagger_{i\bar\sigma}(\tau,\mathbf{r}_1)V_{ij}(\mathbf{r}_1;\mathbf{r}_2)\psi_{j\bar\sigma}(\tau,\mathbf{r}_1)\psi_{j\sigma}(\tau,\mathbf{r}_2) \\ &+ \frac{1}{2}\int \mathrm{d}\mathbf{r}_1 \int \mathrm{d}\mathbf{r}_2 \left(\sum_{i=1,\sigma}^{2} \psi^\dagger_{i\sigma}(\tau,\mathbf{r}_1)\psi_{i\sigma}(\tau,\mathbf{r}_1) - n\right) \\ &\times V_c(\mathbf{r}_1 - \mathbf{r}_2)\left(\sum_{j=1,\sigma'} \psi^\dagger_{j\sigma'}(\tau,\mathbf{r}_2)\psi_{j\sigma'}(\tau,\mathbf{r}_2) - n\right). \end{aligned} \tag{2}$$

Here $\psi_{i\sigma}(\tau,\mathbf{r})$ is a fermion field with the spin $\sigma = \uparrow, \downarrow$, $\bar\sigma \equiv -\sigma$, $i,j = 1,2$ is the band index, $\varepsilon_i(\mathbf{k}) = \mathbf{k}^2/2m_i$ is the dispersion law in ith band with the effective mass of the carriers m_i, τ is the imaginary time and $V_{ij}(\mathbf{r}_1;\mathbf{r}_2) = V_{ij}\delta(\mathbf{r}_1 - \mathbf{r}_2)$ is an attractive short-range potential, $V_c(\mathbf{r}_1 - \mathbf{r}_2)$ is the long range Coulomb interaction, n is the neutralizing background charge density. Throughout the paper we call the superconducting system *neutral* if the last term of equation (2) is omitted and *charged* if this term is taken into account. Even in the latter case the whole superconductor remains, of course, neutral due to the neutralizing ionic background. Throughout the paper $\hbar = k_B = 1$ units are chosen.

Introducing Nambu variables

$$\Psi_i(\tau,\mathbf{r}) = \begin{pmatrix} \psi_{i\uparrow}(\tau,\mathbf{r}) \\ \psi^\dagger_{i\downarrow}(\tau,\mathbf{r}) \end{pmatrix},$$
$$\Psi^\dagger_i(\tau,\mathbf{r}) = \begin{pmatrix} \psi^\dagger_{i\uparrow}(\tau,\mathbf{r}) & \psi_{i\downarrow}(\tau,\mathbf{r}) \end{pmatrix} \tag{3}$$

we rewrite the action as a sum

$$S = S_0 + S_{pair} + S_C \tag{4}$$

of free

$$S_0 = -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}\mathbf{r} \sum_{i=1}^{2} \Psi^\dagger_i(x)[\hat I \partial_\tau + \tau_3(\varepsilon_i(-\mathrm{i}\nabla) - \mu)]\Psi_i(x), \tag{5}$$

pairing

$$S_{pair} = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}\mathbf{r} \sum_{i,j=1}^{2} V_{ij}\Psi^\dagger_i \tau_+ \Psi_i(x)\Psi^\dagger_j \tau_- \Psi_j(x), \tag{6}$$

and Coulomb

$$\begin{aligned} S_C = -\frac{1}{2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}\mathbf{r}_1 \int \mathrm{d}\mathbf{r}_2 &\left(\sum_{i=1}^{2} \Psi^\dagger_i(x)\tau_3\Psi_i(x) - n\right) \\ \times V_c(\mathbf{r}_1 - \mathbf{r}_2) &\left(\sum_{j=1}^{2} \Psi^\dagger_j(x)\tau_3\Psi_j(x) - n\right) \end{aligned} \tag{7}$$

parts. Here $\tau_\pm = (\tau_1 \pm \mathrm{i}\tau_2)/2$, τ_λ ($\lambda = 1,2,3$) are Pauli matrices.

The easiest way to treat S_{pair} is to introduce Hubbard-Stratonovich fields Φ_i for each band, so that

$$\begin{aligned} S_{pair}(\Phi_i,\Phi^*_i,\Psi_i,\Psi^\dagger_i) = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}\mathbf{r}\Big[&- g_{11}|\Phi_1(x)|^2 \\ &- g_{22}|\Phi_2(x)|^2 + g_{12}(\Phi^*_1(x)\Phi_2(x) + \Phi^*_2(x)\Phi_1(x)) \\ &+ \sum_{i=1}^{2} g_{ii}(\Phi^*_i(x)\Psi^\dagger_i\tau_-\Psi_i(x) + \Phi_i(x)\Psi^\dagger_i(x)\tau_+\Psi_i(x))\Big], \end{aligned} \tag{8}$$

where the coupling constants g_{ij} are expressed in terms of the original constants V_{ij}

$$g_{11} = V_{11}\left(1 - \frac{V_{12}^2}{V_{11}V_{22}}\right), \quad g_{22} = V_{22}\left(1 - \frac{V_{12}^2}{V_{11}V_{22}}\right),$$
$$g_{12} = V_{12}\left(1 - \frac{V_{12}^2}{V_{11}V_{22}}\right). \tag{9}$$

For the repulsion part only one Hubbard-Stratonovich field φ is necessary:

$$S_C(\varphi, \Psi_i, \Psi_i^\dagger) = \int d\tau \int d\mathbf{r}_1 \int d\mathbf{r}_2 \bigg[-\frac{1}{2} e\varphi(\tau,\mathbf{r}_1) V_c^{-1}(\mathbf{r}_1-\mathbf{r}_2) e\varphi(\tau,\mathbf{r}_2) + ie\varphi(\tau,\mathbf{r}_1)\left(\sum_{i=1}^{2}\Psi_i^\dagger(x)\tau_3\Psi_i(x) - n\right)\delta(\mathbf{r}_1-\mathbf{r}_2)\bigg]. \quad (10)$$

Thus the partition function of the system can be presented as

$$Z = \int \mathcal{D}\Phi_i \mathcal{D}\Phi_i^* \mathcal{D}\varphi \mathcal{D}\Psi_i \mathcal{D}\Psi_i^\dagger \exp\left[S_0\left(\Psi_i, \Psi_i^\dagger\right) + S_{pair}\left(\Phi_i, \Phi_i^*, \Psi_i, \Psi_i^\dagger\right) + S_C\left(\varphi, \Psi_i, \Psi_i^\dagger\right)\right]. \quad (11)$$

3 Effective potential in the modulus-phase variables

3.1 Modulus-phase variables

The modulus-phase variables Δ_i and θ_i in two bands are introduced exactly as discussed in [7]:

$$\Phi_i(\tau,\mathbf{r}) = \Delta_i(\tau,\mathbf{r})\exp[i\theta_i(\tau,\mathbf{r})],$$
$$\Psi_i(\tau,\mathbf{r}) = \begin{pmatrix} e^{i\theta_i(\tau,\mathbf{r})/2} & 0 \\ 0 & e^{-i\theta_i(\tau,\mathbf{r})/2}\end{pmatrix}\Upsilon_i(\tau,\mathbf{r}), \quad (12)$$

making the moduli Δ_i of the pairing field real. We note that since we restricted our analysis to $T < T_c$ in zero external field, so that no vortices are present, the phases $\theta_i(\tau,\mathbf{r})$ are non-singular, so that the transformations (12) do not change the magnetic field "seen" by quasiparticles. Now absorbing g_{11} and g_{22} in Δ_1 and Δ_2 and integrating out neutral fermions Υ_i we obtain (see *e.g.* [7,15,19])

$$Z = \int \Delta_i \mathcal{D}\Delta_i \mathcal{D}\theta_i \mathcal{D}\varphi \exp[-\beta\Omega(\Delta_i,\theta_i,\varphi)] \quad (13)$$

with the effective thermodynamical potential

$$\beta\Omega(\Delta_i,\theta_i,\varphi) = \int_0^\beta d\tau d\mathbf{r}\left[\frac{\Delta_1^2}{g_{11}} + \frac{\Delta_2^2}{g_{22}} - \frac{2g_{12}}{g_{11}g_{22}}\Delta_1\Delta_2\cos(\theta_1-\theta_2)\right] + \int_0^\beta d\tau d\mathbf{r}_1 d\mathbf{r}_2 \bigg[\frac{1}{2}e\varphi(\tau,\mathbf{r}_1)V_c^{-1}(\mathbf{r}_1,\mathbf{r}_2)e\varphi(\tau,\mathbf{r}_2) + ie\varphi(\tau,\mathbf{r}_1)n\delta(\mathbf{r}_1-\mathbf{r}_2)\bigg] - \mathrm{Tr\ Ln}G_1^{-1} - \mathrm{Tr\ Ln}G_2^{-1}, \quad (14)$$

where the Green's function

$$G_i^{-1} \equiv \mathcal{G}_i^{-1} - \Sigma_i(\partial\theta_i) = -\hat{I}\partial_\tau + \tau_3\left(\frac{\nabla^2}{2m_i} + \mu\right) + \tau_1\Delta_i(\tau,\mathbf{r}) - \Sigma_i(\partial\theta_i) \quad (15)$$

with

$$\Sigma_i(\partial\theta_i) \equiv \tau_3\left[\frac{i\partial_\tau\theta_i}{2} - ie\varphi(\tau,\mathbf{r}) + \frac{(\nabla\theta_i)^2}{8m_i}\right] - \hat{I}\left[\frac{i\nabla^2\theta_i}{4m_i} + \frac{i\nabla\theta_i(\tau,\mathbf{r})\nabla}{2m_i}\right] \quad (16)$$

that depends only on the time and space derivatives of θ_i, but not on the phase θ_i itself. Then we can represent Ω as the sum

$$\Omega(\Delta_i,\theta_i,\varphi) \simeq \Omega_{\mathrm{kin}}(\Delta_i,\partial\theta_i,\varphi) + \Omega_{\mathrm{pot}}(\Delta_i,\theta_i,\varphi), \quad (17)$$

where

$$\Omega_{\mathrm{kin}}(\Delta_i,\partial\theta_i) = T\sum_{i=1}^{2}\mathrm{Tr}\sum_{n=1}^{\infty}\frac{1}{n}(\mathcal{G}_i\Sigma_i)^n\bigg|_{\Delta_i=\mathrm{const.}} \quad (18)$$

is the sum of the energies of the phase fluctuations in each band and

$$\Omega_{\mathrm{pot}}(\Delta_i,\theta_i,\varphi) = \left(\frac{\Delta_1^2}{g_{11}} + \frac{\Delta_2^2}{g_{22}} - \frac{2g_{12}}{g_{11}g_{22}}\Delta_1\Delta_2\cos(\theta_1-\theta_2) - \mathrm{Tr\ Ln}\mathcal{G}_1^{-1} - \mathrm{Tr\ Ln}\mathcal{G}_2^{-1}\right)\bigg|_{\Delta_i=\mathrm{const.}} + \int_0^\beta d\tau d\mathbf{r}_1 d\mathbf{r}_2 \times\bigg[\frac{1}{2}e\varphi(\tau,\mathbf{r}_1)V_c^{-1}(\mathbf{r}_1,\mathbf{r}_2)e\varphi(\tau,\mathbf{r}_2) + ie\varphi(\tau,\mathbf{r}_1)n\delta(\mathbf{r}_1-\mathbf{r}_2)\bigg]. \quad (19)$$

In Ω_{pot} the most important for the appearance of Leggett's mode term is the Josephson coupling energy of the condensates in two bands. This term explicitly depends on the relative $\theta_1-\theta_2$ phase of two condensates.

3.2 The system of gap equations

As pointed out in [14] while V_{11} and V_{22} are completely fixed by the physics of the problem, the phase of V_{12} is a matter of convention (corresponding to the choice of the relative phases of the Bloch waves in the two bands). As in [14] we choose V_{12} to be real and positive. In this case the condition of minima of Ω_{pot} with respect to $\theta_1-\theta_2$ gives $\theta_1 = \theta_2$, so that the system of the gap equations $\partial\Omega_{pot}/\partial\Delta_i = 0$ has the form

$$\begin{cases}\Delta_1 - \frac{g_{12}}{g_{22}}\Delta_2 - \Delta_1 g_{11}N_1F(\Delta_1) = 0,\\ \Delta_2 - \frac{g_{12}}{g_{11}}\Delta_1 - \Delta_2 g_{22}N_2F(\Delta_2) = 0,\end{cases} \quad (20)$$

$$\mathcal{M}_i^{-1} = \begin{bmatrix} -\Omega_n^2 {}^i\Pi_{33}(K) + {}^i\Lambda^{\alpha\beta}(K)K_\alpha K_\beta - \mathrm{i}\Omega_n K_\alpha {}^i\Pi_{03}^\alpha(K) - \mathrm{i}\Omega_n K_\alpha {}^i\Pi_{30}^\alpha(K) & 2\mathrm{i}\Omega_n {}^i\Pi_{33}(K) - 2K_\alpha {}^i\Pi_{30}^\alpha(K) \\ -2\mathrm{i}\Omega_n {}^i\Pi_{33}(K) + 2K_\alpha {}^i\Pi_{30}^\alpha(K) & -4 {}^i\Pi_{33}(K) \end{bmatrix} \quad (24)$$

where $N_i = m_i p_{Fi}/(2\pi^2)$ is the density of states (per spin) in ith band (p_{Fi} is the Fermi momentum) and

$$F(\Delta_i) = \int_0^{\omega_D} \frac{\mathrm{d}\xi}{\sqrt{\xi^2+\Delta_i^2}} \tanh\frac{\sqrt{\xi^2+\Delta_i^2}}{2T} \quad (21)$$

with the Debye frequency, ω_D which is for simplicity assumed to be the same in each band.

This system of the equation can be transformed to the standard form derived by Moskalenko [20] and Suhl *et al.* [21]

$$\begin{cases} \Delta_1[1 - V_{11}N_1F(\Delta_1)] = \Delta_2 V_{12} N_2 F(\Delta_2), \\ \Delta_2[1 - V_{22}N_2F(\Delta_2)] = \Delta_1 V_{12} N_1 F(\Delta_1). \end{cases} \quad (22)$$

It appears, however, that in contrast to Leggett's approach [14] there is no need of explicit use of the system (22) to obtain the spectrum of collective excitations. In what follows we assume that the values of the superconducting gaps $\Delta_1(T)$ and $\Delta_2(T)$ can, in principle, be determined from the system (22) or taken directly from the experiment.

3.3 Effective potential for the collective modes

Since the values of the gaps are fixed, *i.e.* there are no amplitude fluctuations that is reasonable for $T \ll T_c$ it is sufficient to consider only the part of effective potential depending on θ_i and φ.

Following the same route as described, for example, in references [9,10,15], we arrive at the following frequency-momentum representation

$$\beta\Omega\{\theta_i,\varphi\} = \frac{T}{8}\sum_{n=-\infty}^{\infty}\int\frac{\mathrm{d}\mathbf{K}}{(2\pi)^3}\Bigg(\varphi(-K)4e^2V_c^{-1}(\mathbf{K})\varphi(K) + \frac{8g_{12}}{g_{11}g_{22}}\Delta_1\Delta_2(\theta_1(-K)-\theta_2(-K))(\theta_1(K)-\theta_2(K)) + \sum_{i=1}^{2}\left[\theta_i(-K) \;\; e\varphi(-K)\right]\mathcal{M}_i^{-1}\begin{bmatrix}\theta_i(K)\\ e\varphi(K)\end{bmatrix}\Bigg), \quad (23)$$

where the Josephson term was expanded up to quadratic term and unimportant constant was dropped out. The matrix $\mathcal{M}_i$ in (23) is

see equation (24) above

and we introduced short-hand notations $K = (\mathrm{i}\Omega_n, \mathbf{K})$ with $\Omega_n = 2\pi nT$ and $\mathbf{K}$ being 3D vector (summation over dummy indices $\alpha,\beta = 1,2,3$ is implied). In equation (24) ${}^i\Lambda^{\alpha\beta} = {}^i\Lambda_0^{\alpha\beta} + {}^i\Pi_{00}^{\alpha\beta}$ is the bare (unrenormalized by the phase fluctuations) superfluid stiffness, where the current-current polarization function, ${}^i\Pi_{00}$, is

$${}^i\Pi_{00}^{\alpha\beta}(\mathrm{i}\Omega_n,\mathbf{K}) \equiv T\sum_{l=-\infty}^{\infty}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\,{}^i\pi_{00}(\mathrm{i}\Omega_n,\mathbf{K};\mathrm{i}\omega_l,\mathbf{k})v_{Fi\alpha}(\mathbf{k})v_{Fi\beta}(\mathbf{k}) \quad (25)$$

with the Fermi velocity $v_{Fi\alpha}(\mathbf{k}) = \partial\xi_i(\mathbf{k})/\partial k_\alpha|_{k=k_{Fi}}$ (here $\xi_i(\mathbf{k}) = \varepsilon_i(\mathbf{k}) - \mu$); the density-density polarization function, ${}^i\Pi_{33}$, is

$${}^i\Pi_{33}(\mathrm{i}\Omega_n,\mathbf{K}) \equiv T\sum_{l=-\infty}^{\infty}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\,{}^i\pi_{33}(\mathrm{i}\Omega_n,\mathbf{K};\mathrm{i}\omega_l,\mathbf{k}) \quad (26)$$

and the density-current polarization function, ${}^i\Pi_{03}^\alpha$ is

$${}^i\Pi_{03}^{\alpha}(\mathrm{i}\Omega_n,\mathbf{K}) \equiv T\sum_{l=-\infty}^{\infty}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\,{}^i\pi_{03}(\mathrm{i}\Omega_n,\mathbf{K};\mathrm{i}\omega_l,\mathbf{k})v_{Fi\alpha}(\mathbf{k}). \quad (27)$$

${}^i\pi_{\lambda\kappa}$ in equations (25–27) is given by

$${}^i\pi_{\lambda\kappa}(\mathrm{i}\Omega_n,\mathbf{K};\mathrm{i}\omega_l,\mathbf{k}) \equiv \mathrm{tr}[{}^i\mathcal{G}(\mathrm{i}\omega_l+\mathrm{i}\Omega_n,\mathbf{k}+\mathbf{K}/2)\tau_\lambda {}^i\mathcal{G}(\mathrm{i}\omega_l,\mathbf{k}-\mathbf{K}/2)\tau_\kappa], \quad (\tau_0 \equiv \hat{I}), \quad (28)$$

where the neutral fermion Green's function coincides with the usual Green's function of the BCS theory

$${}^i\mathcal{G}(\mathrm{i}\omega_n,\mathbf{k}) = -\frac{\mathrm{i}\omega_n\hat{I} + \tau_3\xi_i(\mathbf{k}) - \tau_1\Delta_i}{\omega_n^2+\xi_i^2(\mathbf{k})+\Delta_i^2}, \quad \omega_n = \pi(2n+1)T. \quad (29)$$

${}^i\Lambda_0^{\alpha\beta}$ above is the first order contribution to the superfluid stiffness:

$${}^i\Lambda_0^{\alpha\beta} = \int\frac{\mathrm{d}^3k}{(2\pi)^2}n_i(\mathbf{k})m_i^{-1}\delta_{\alpha\beta} = \frac{n_i}{m_i}\delta_{\alpha\beta}, \quad (30)$$

with

$$n_i(\mathbf{k}) = 1 - \frac{\xi_i(\mathbf{k})}{E_i(\mathbf{k})}\tanh\frac{E_i(\mathbf{k})}{2T}, \quad E_i(\mathbf{k}) = \sqrt{\xi_i^2(\mathbf{k})+\Delta_i^2}. \quad (31)$$

Writing equation (23) we omitted the time derivative term linear in the phase (see *e.g.* [10,11]) which is irrelevant for the present analysis.

For the purpose of completeness we give the explicit expressions for the polarizations (25–27) in Appendix A. These expressions are important if, for example, one is interested in the damping of the collective modes. In what follows we consider these polarizations in the hydrodynamic limit, $\Omega_n = 0$ and $\mathbf{K} \to 0$ at $T = 0$. In this case ${}^i\Pi_{33}(0,\mathbf{0}) = -2N_i$, ${}^i\Lambda^{\alpha\beta}(0,\mathbf{0}) = {}^i\Lambda_0^{\alpha\beta} = n_i/m_i = 2N_i v_{Fi}^2/3$ and ${}^i\Pi_{03}^{\alpha}(0,\mathbf{0}) = 0$. Recall that N_i is the density of states per spin, while in [14] the density of states per particle was used.

As was mentioned above, calculating the values ${}^i\Pi_{33}(0,\mathbf{0})$ and ${}^i\Lambda^{\alpha\beta}(0,\mathbf{0})$ there is no need to substitute the gap equations (22) since the coupling constants V_{ij} *do not enter* ${}^i\Pi_{33}(0,\mathbf{0})$ and ${}^i\Lambda^{\alpha\beta}(0,\mathbf{0})$, but enter only in the Josephson coupling term. By contrast, the original derivation of Leggett [14] (see after Eq. (3.8)) explicitly uses the system (22). Note that in a more simple case of one band system the Josephson term is absent and one immediately gets the BAG mode without referring to the gap equation [8].

We note also that while ${}^i\Pi_{03}^{\alpha}$ is zero for the case considered in the present paper, this term is crucial for the existence of Carlson-Goldman mode [13,15].

Having the general representation for the thermodynamical potential $\Omega\{\theta_i,\varphi\}$ we are ready to obtain the spectrum of collective excitations. We start with a more simple case of a neutral superconductor.

4 Collective excitations in a neutral superconductor

To consider neutral superconductor one can formally set $e = 0$ in (23), so that the terms with the electric potential φ disappear from the equations and we arrive at

$$\beta\Omega\{\theta_i\} = \frac{T}{8}\sum_{n=-\infty}^{\infty}\int\frac{d\mathbf{K}}{(2\pi)^3}\left(\frac{8V_{12}}{V_{11}V_{22}-V_{12}^2}\times\Delta_1\Delta_2(\theta_1(-K)-\theta_2(-K))(\theta_1(K)-\theta_2(K)) + \sum_{i=1}^{2}\theta_i(-K)M_{\theta i}^{-1}\theta_i(K)\right), \tag{32}$$

where

$$M_{\theta i}^{-1} = -\,\Omega_n^2\,{}^i\Pi_{33}(K) + {}^i\Lambda^{\alpha\beta}(K)K_\alpha K_\beta - \mathrm{i}\Omega_n K_\alpha{}^i\Pi_{03}^{\alpha}(K) - \mathrm{i}\Omega_n K_\alpha{}^i\Pi_{30}^{\alpha}(K). \tag{33}$$

As was already mentioned, we consider M_{θ_i} in hydrodynamic ($\Omega_n = 0$, $\mathbf{K} \to 0$) limit at $T = 0$:

$$M_{\theta_i}^{-1} = 2N_i(\Omega_n^2 + c_i^2K^2), \tag{34}$$

where $c_i^2 = v_{Fi}^2/3$ is the velocity of the BAG mode in ith band, so that equation (32) becomes

$$\beta\Omega\{\theta_i\} = \frac{T}{8}\sum_{n=-\infty}^{\infty}\int\frac{d\mathbf{K}}{(2\pi)^3}\begin{bmatrix}\theta_1(-K) & \theta_2(-K)\end{bmatrix}\Theta^{-1}\begin{bmatrix}\theta_1(K)\\ \theta_2(K)\end{bmatrix}, \tag{35}$$

with

$$\Theta^{-1} = \begin{bmatrix} M_{\theta_1}^{-1} + A & -A \\ -A & M_{\theta_2}^{-1} + A\end{bmatrix}, \quad A \equiv \frac{8V_{12}\Delta_1\Delta_2}{V_{11}V_{22}-V_{12}^2}\,. \tag{36}$$

Finally, solving the equation $\det\Theta^{-1} = 0$ for collective modes and making an analytical continuation $\mathrm{i}\Omega_n \to \omega + \mathrm{i}0$ we arrive at

$$\omega^2 = \frac{1}{2}\left[\omega_0^2 + (c_1^2+c_2^2)K^2 \pm\sqrt{\omega_0^4 + (c_1^2-c_2^2)^2K^4 - 2\omega_0^2\frac{N_1-N_2}{N_1+N_2}(c_1^2-c_2^2)K^2}\right] \tag{37}$$

with

$$\omega_0^2 = \frac{N_1+N_2}{2N_1N_2}\frac{8V_{12}\Delta_1\Delta_2}{V_{11}V_{22}-V_{12}^2}\,. \tag{38}$$

(For a direct comparison with [14] recall that the density of states used here is twice less.) In the limit $K \to 0$ ($v_{Fi}K \ll \omega_0$) one obtains from (37)

$$\omega^2 = c^2K^2, \quad c^2 = \frac{N_1c_1^2 + N_2c_2^2}{N_1+N_2} \quad \text{for “–”;}$$
$$\omega^2 = \omega_0^2 + v^2K^2, \quad v^2 = \frac{N_1c_2^2 + N_2c_1^2}{N_1+N_2} \quad \text{for “+”.} \tag{39}$$

The first solution of (39) corresponds to BAG mode, while the second solution is Leggett's mode. This collective mode is only possible if $\omega_0^2 > 0$. Since $V_{12} > 0$ this implies that Leggett's mode exists for $V_{11}V_{22} - V_{12}^2 > 0$.

5 Collective excitations in a charged superconductor

As was discussed in Introduction the long-distance Coulomb interaction has a drastic influence on the BAG

Table 1. The estimate of the frequency ω_0. The values of the superconducting $\Delta_1 = 1.8$ meV and $\Delta_2 = 6.8$ meV from [28] are used.

Reference	λ_{11}	λ_{22}	λ_{12}	λ_{21}	ω_0, meV	$\omega_0/2\Delta_1$
Liu *et al.* [24] and Barabash [26]	0.96	0.28	0.16	0.22	8.9	2.5
Golubov *et al.* [25]	1.017	0.448	0.213	0.155	6.5	1.8

mode transforming it in the plasma mode. Here we consider how Leggett's mode is affected by the Coulomb interaction. Let us rewrite action (23) in the hydrodynamic limit as one matrix

$$\beta\Omega\{\theta_i,\varphi\} = \frac{T}{8}\sum_{n=-\infty}^{\infty}\int\frac{\mathrm{d}\mathbf{K}}{(2\pi)^3}\theta_1(-K)\ \theta_2(-K) \times e\varphi(-K)\mathcal{M}^{-1}\begin{bmatrix}\theta_1(K)\\ \theta_2(K)\\ e\varphi(K)\end{bmatrix}, \quad (40)$$

where

$$\mathcal{M}^{-1} = \begin{bmatrix} M_{\theta_1}^{-1} + A & -A & -4\mathrm{i}\Omega_n N_1 \\ -A & M_{\theta_2}^{-1} + A & -4\mathrm{i}\Omega_n N_2 \\ 4\mathrm{i}\Omega_n N_1 & 4\mathrm{i}\Omega_n N_2 & 4(2N_1 + 2N_2 + V_c^{-1}(\mathbf{K})) \end{bmatrix} \quad (41)$$

and $M_{\theta_1}^{-1}$ and A are given by equations (34) and (36), respectively. As one can readily see, setting $e = 0$ in equation (40) we reduce the action (40) with 3×3 matrix $\mathcal{M}^{-1}$ that describes the phase fluctuations in the charged system to the action (35) with 2×2 matrix Θ^{-1} that describes the collective excitations in a neutral system.

Again the spectrum of collective excitations can be found solving the equation $\det\mathcal{M}^{-1} = 0$. Neglecting the inverse of Coulomb interaction in the limit $\mathbf{K} \to 0$ ($V_c^{-1}(\mathbf{K}) = \mathbf{K}^2/(4\pi e^2) \to 0$), we obtain

$$\Omega^2 = \omega_0^2 + v^2K^2, \qquad v^2 = \frac{(N_1+N_2)c_1^2c_2^2}{N_1c_1^2 + N_2c_2^2}\cdot \quad (42)$$

Since the inverse Coulomb potential $\sim K^2$ is neglected, the equation for the collective modes has the only solution describing Leggett's mode. Comparing (39) and (42) one can see that in accordance with [14] the value of ω_0 is insensitive to the Coulomb interaction but the velocity v is affected. It is interesting that the value of v itself does not depend on the explicit form of $V_c(\mathbf{K})$.

Including the Coulomb interaction $V_c(\mathbf{K})$ one can also check that the equation $\det\mathcal{M}^{-1} = 0$ has a plasma mode as a solution. Indeed considering the simplest $V_{12} = 0$ ($A = 0$) case we obtain the plasma frequency for in-phase oscillations in both bands

$$\Omega_p^2 = 8\pi e^2(N_1c_1^2 + N_2c_2^2) = 4\pi e^2\left(\frac{n_1}{m_1} + \frac{n_2}{m_2}\right)\cdot \quad (43)$$

6 Implications for MgB_2

There are currently many indications that the recently discovered MgB_2 superconductor [17] can be described by the classical two-gap model [20,21] which convincingly fits the specific heat [22] and penetration depth measurements [23]. In particular, in [24–26] even the values of the coupling constants for the system of equations (22) for the two-gap and two-band model of MgB_2 were given.

We note that one of the alternative explanations of the observed anisotropy of the upper critical field uses a model with the anisotropic *s*-wave pairing [27] which is also rather close to the two-gap and two-band scenario.

To be experimentally observable Leggett's mode should have the value of ω_0 in (42) well separated from the two-particle threshold given by the smallest gap, Δ_1. Here we estimate the value of ω_0 using recently suggested values of the coupling constants that enter the system of equations (22). Introducing the dimensionless coupling constants, $\lambda_{ij} = N_iV_{ij}$ that are often used for the description of the two-band model, we may rewrite equation (38) in the following form

$$\omega_0^2 = \frac{\lambda_{12}+\lambda_{21}}{\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}}4\Delta_1\Delta_2. \quad (44)$$

Our estimates of ω_0 are summarized in Table 1.

They show that for the values of the two-band model parameters known at present for the two-band model of MgB_2, Leggett's mode arises above the two-particle threshold, $\omega_0/2\Delta_1 > 1$, and unlikely to be observed. We do not exclude, however, that Leggett's mode can be observed in MgB_2 if the values of the coupling constants are revised, so that the interband coupling constants λ_{12} and λ_{21} would become smaller allowing $\omega_0/2\Delta_1 < 1$. The observation of Leggett's mode would provide an additional insight to the underlying physics of this superconductor.

7 Conclusion

To conclude, we readdressed the collective excitations of the relative phase of the two condensates in a clean two-band superconductor using the effective "phase action" formalism. This formalism has proved itself as a convenient and economic method of the investigation of the collective modes in superconductors. Our estimates of the lowest frequency of Leggett's mode for MgB_2 show that it can hardly be observed in this superconductor.

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Appendix A: The expressions for polarizations Π_{ij}

The expressions for the polarization functions $\Pi_{\lambda\kappa}$ (for simplicity we omit the band index i) are (see *e.g.* [9,10])

$$\begin{bmatrix} \Pi_{00}^{\alpha\beta}(\mathrm{i}\Omega_n, \mathbf{K}) \\ \Pi_{33}(\mathrm{i}\Omega_n, \mathbf{K}) \end{bmatrix} = -\int \frac{\mathrm{d}^3k}{(2\pi)^3} \Bigg\{ \frac{1}{2}\left(1 - \frac{\xi_-\xi_+ \pm \Delta^2}{E_-E_+}\right) \times \left[\frac{1}{E_+ + E_- + \mathrm{i}\Omega_n} + \frac{1}{E_+ + E_- - \mathrm{i}\Omega_n}\right] \times [1 - n_F(E_-) - n_F(E_+)] + \frac{1}{2}\left(1 + \frac{\xi_-\xi_+ \pm \Delta^2}{E_-E_+}\right)\left[\frac{1}{E_+ - E_- + \mathrm{i}\Omega_n} + \frac{1}{E_+ - E_- - \mathrm{i}\Omega_n}\right][n_F(E_-) - n_F(E_+)]\Bigg\} V_\pm(\mathbf{k}),$$

$$V_\pm(\mathbf{k}) \equiv \begin{bmatrix} v_{F\alpha}(\mathbf{k}) v_{F\beta}(\mathbf{k}), & \text{"+"}; \\ 1, & \text{"-"}. \end{bmatrix}, \quad \text{(A.1)}$$

and

$$\Pi_{03}^{\alpha}(\mathrm{i}\Omega_n, \mathbf{K}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \Bigg\{ \left(\frac{\xi_+}{2E_+} - \frac{\xi_-}{2E_-}\right) \times \left[\frac{1}{E_+ + E_- + \mathrm{i}\Omega_n} - \frac{1}{E_+ + E_- - \mathrm{i}\Omega_n}\right] \times [1 - n_F(E_-) - n_F(E_+)] + \left(\frac{\xi_+}{2E_+} + \frac{\xi_-}{2E_-}\right) \times \left[\frac{1}{E_+ - E_- + \mathrm{i}\Omega_n} - \frac{1}{E_+ - E_- - \mathrm{i}\Omega_n}\right] \times [n_F(E_-) - n_F(E_+)]\Bigg\} v_{F\alpha}(\mathbf{k}), \quad \text{(A.2)}$$

where $\xi_\pm \equiv \xi(\mathbf{k} \pm \mathbf{K}/2)$, $E_\pm \equiv E(\mathbf{k} \pm \mathbf{K}/2)$. One can also check that $\Pi_{30}^{\alpha}(\mathrm{i}\Omega_n, \mathbf{K}) = \Pi_{03}^{\alpha}(\mathrm{i}\Omega_n, \mathbf{K})$. The second term in equations (A.1) and (A.2) gives the contribution of the thermally excited quasiparticles (*i.e.* "normal" fluid component). This is the term responsible for the appearance of the *Landau terms* in the effective action [9,10].

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